(a) $\int_{-1}^{1} \frac{1}{x^{2/3}}\,dx$ ← improper since $1/x^{2/3}$ is not defined at $x=0$.

$$= \underbrace{\int_{-1}^{0} \frac{1}{x^{2/3}}\,dx}_{} + \underbrace{\int_{0}^{1} \frac{1}{x^{2/3}}\,dx}_{}$$

Both integrals must exists as a real number (convergent) for the original integral to exits as a real number (convergent)

$$\int_{-1}^{0} \frac{1}{x^{2/3}}\,dx = \lim_{a\to 0^-} \int_{-1}^{a} x^{-2/3}\,dx$$

$$= \lim_{a\to 0^-} \left[\frac{x^{1/3}}{1/3} \right]_{-1}^{a} = \lim_{a\to 0^-} \left[3x^{1/3} \right]_{-1}^{a}$$

$$= \lim_{a\to 0^-} \left[3a^{1/3} - 3(-1)^{1/3} \right] = 0 - 3(-1) = 3$$

$$\int_{0}^{1} \frac{1}{x^{2/3}}\,dx = \lim_{b\to 0^+} \int_{b}^{1} x^{-2/3}\,dx = \lim_{b\to 0^+} \left[3x^{1/3} \right]_{b}^{1}$$

$$= \lim_{b\to 0^+} \left[3(1)^{1/3} - 3b^{1/3} \right] = 3(1) - 0 = 3$$

$$\int_{-1}^{1} \frac{1}{x^{2/3}}\,dx = 3+3 = 6 \text{ (is convergent)}$$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{16+x^2}\,dx$ ← improper since the interval of integration is infinite in length

$$= \int_{-\infty}^{0} \frac{1}{16+x^2}\,dx + \int_{0}^{\infty} \frac{1}{16+x^2}\,dx$$

$$\int_{-\infty}^{0} \frac{1}{16+x^2}\,dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{16+x^2}\,dx$$

$$= \lim_{a\to-\infty} \frac{1}{16}\int_{a}^{0} \frac{1}{1+\frac{x^2}{16}}\,dx$$

$$= \lim_{a\to-\infty} \frac{1}{16}\int_{a}^{0} \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx$$

$u = \frac{x}{4}$

$du = \frac{1}{4}\,dx$

$dx = 4du$

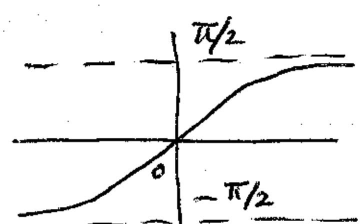
$$= \lim_{a\to-\infty} \frac{1}{\cancel{16}_{4}}\int_{a/4}^{0} \frac{1}{1+u^2}\cdot \cancel{4}du$$

$$= \lim_{a\to\infty} \frac{1}{4}\Big[\arctan u\Big]_{a/4}^{0}$$

$$= \lim_{a\to\infty} \frac{1}{4}\Big[\underbrace{\arctan(0)}_{0} - \underbrace{\arctan\left(\tfrac{a}{4}\right)}_{-\pi/2}\Big] \qquad \tfrac{a}{4}\to-\infty$$

2

$$= \frac{1}{4}\left[-\left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{8}.$$



$$\int_0^\infty \frac{1}{16+x^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{16+x^2}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{16} \int_0^b \frac{1}{1+\frac{x^2}{16}}\,dx = \lim_{b\to\infty} \frac{1}{16} \int_0^b \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx$$

$$u = \frac{x}{4}$$

$$du = \frac{1}{4}dx$$

$$= \lim_{b\to\infty} \frac{1}{\cancel{16}_{\,4}} \int_0^{b/4} \frac{1}{1+u^2} \cdot \cancel{4}\,du$$

$$= \lim_{b\to\infty} \frac{1}{4} \Big[\arctan(u) \Big]_0^{b/4}$$

$$= \lim_{b\to\infty} \frac{1}{4} \Big[\underbrace{\arctan\left(\overset{\to\infty}{\frac{b}{4}}\right)}_{\to \pi/2} - \underbrace{\cancel{\arctan(0)}}_{0} \Big]$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right) = \pi/8$$

$$\int_{-\infty}^\infty \frac{1}{16+x^2}\,dx = \frac{\pi}{8} + \frac{\pi}{8} = \boxed{\frac{\pi}{4}}$$

(c) $\int_0^\infty xe^{-x}\,dx = \lim_{a\to\infty} \int_0^a xe^{-x}\,dx$

Apply int. by parts.

$u = x$
$du = dx$
$dv = e^{-x}dx$
$v = -e^{-x}$

$= \lim_{a\to\infty} \left[\left[-xe^{-x}\right]_0^a - \int_0^a -e^{-x}\,dx \right]$

$= \lim_{a\to\infty} \left[-ae^{-a} + 0 + \int_0^a e^{-x}\,dx \right]$

$= \lim_{a\to\infty} \left[-\frac{a}{e^a} + \left[-e^{-x}\right]_0^a \right]$

$= \lim_{a\to\infty} \left[-\frac{a}{e^a} + \left[-e^{-a} + e^0\right] \right]$

$= \lim_{a\to\infty} \left[\frac{-a}{e^a} - \frac{1}{e^a} + 1 \right]$

Hopital's Rule : $\lim_{a\to\infty} \frac{a}{e^a} = \lim_{a\to\infty} \frac{1}{e^a} = 0$

So $\int_0^\infty xe^{-x}\,dx = -0 - 0 + 1 = \boxed{1}$.